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Prevalence of hyperbolicity for complex singular cycles

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—Dedicated to the memory of Professor R. Mañé.

Abstract. In this article we consider two kinds of complex singular cycles arising for vector fields defined on three-dimensional manifolds. We prove that, under some generic conditions, any one parameter family of vector fields passing through these cycles has the following property : Hyperbolicity is a prevalent phenomena.

1. Introduction

Let M be a C^∞, n-dimensional, compact, connected, boundaryless, riemannian manifold. Let $X \in \mathfrak{X}^r(M)$ be a C^r-vector field on M.

Definition. A cycle for the vector field X is a compact, invariant set $\Gamma \subset M$ formed by:

(i) a finite number of singularities and periodic orbits $\Gamma_0 = \{\sigma_0, \dots, \sigma_n\}$;

(ii) the complement $\Gamma_1 = (\Gamma \backslash \Gamma_0)$ is a set of regular trajectories of the vector field X that satisfies:

$(cc)_1$ for any trajectory $\Upsilon \subset \Gamma_1$, there exists $0 \leq i \leq n$ such that $w(\Upsilon) \subset \sigma_{(i+1) mod (n+1)}$ and $\alpha(\Upsilon) \subset \sigma_i$;

$(cc)_2$ given $0 \leq i \leq n$ there exists a trajectory $\Upsilon \subset \Gamma_1$ such that $w(\Upsilon) \subset \sigma_{(i+1) mod (n+1)}$ and $\alpha(\Upsilon) \subset \sigma_i$.

Here $w(\Upsilon)$ (respectively $\alpha(\Upsilon)$) denotes the w-limit set (respectively the α-limit set) of the trajectory Υ.

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A cycle will be called **singular** if it contains a singularity; **hyperbolic** if all critical elements in Γ are hyperbolic.

In this article we will deal with two kind of 3-dimensional, hyperbolic, singular cycle, $\Gamma \subset M^3$, defined in the following way :

The first one contains a unique singularity, $\sigma_0(X)$, a periodic orbit $\sigma_1(X)$ and trajectories (see figure 1):

$$\Upsilon(X) \subset W^u(\sigma_0(X)) \cap W^s(\sigma_1(X)), \Theta(X) \subset W^u(\sigma_1(X)) \cap W^s(\sigma_0(X)).$$

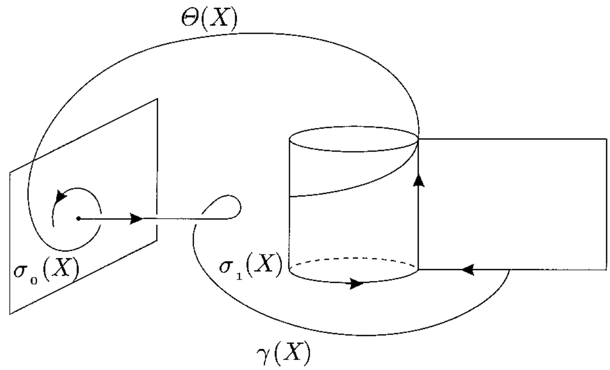

Figure 1.

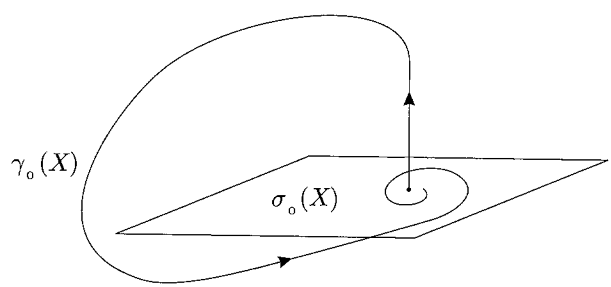

Figure 2.

The second one contains a unique singularity, $\sigma_0(X)$, and a trajectory (see figure 2)

$$\Upsilon(X) \subset W^u(\sigma_0(X)) \cap W^s(\sigma_0(X)) .$$

Let $\mathcal{U}_X \subset \mathfrak{X}^r(M^3)$ be a small neighborhood of X in $\mathfrak{X}^r(M^3)$, with the usual C^r-topology, $r \geq 3$. In the first case we will assume the following regularity conditions :

1. $\Gamma = \{\sigma_0(X), \Upsilon(X), \sigma_1(X), \Theta(X)\}$ where $W_1^u = W_1^u(\sigma_1(X))$ intersects transversally $W_0^s = W^s(\sigma_0(X))$ along the orbit $\Theta(X)$;
2. The eigenvalues of
$$DX(\sigma_0(X)) : T_{\sigma_0(X)}(M^3) \to T_{\sigma_0(X)}(M^3)$$
are $a \pm ib$, and c where $a \leq 0$, $b \neq 0$ and $c \geq 0$, and satisfy a k-Sternberg condition, k big enough to guarantee that we have C^2-linearizing coordinates at $\sigma_0(Y)$, all $Y \in \mathcal{U}_X$, which depends C^2 on Y.
3. Let $P(Y)$ denote the Poincaré map associated to σ_1 at $q_1(Y) \in \sigma_1(Y)$. We suppose that the eigenvalues of $DP_1(q_1(Y))$ are real numbers and satisfy a k-Sternberg condition, k big enough to guarantee that we have C^2- linearizing coordinates which depends C^2 on $Y \in \mathcal{U}_X$, in a neighborhood of $q_1(Y)$.

In the second case we will assume condition 2 above. In this situation interesting dynamics occurs only for
$$-\frac{a(X)}{c(X)} < 1 \, .$$
In the sequel we will assume that we have this condition. A cycle as above is called a *complex expanding singular cycle.*

Given a neighborhood $U \supset \Gamma$ in M^3, let us $\Gamma(Y, U)$ denote the set $\{x \in \bigcap_t Y_t(U) : x \text{ is a chain recurrent point}\}$, for $Y \in \mathcal{U}_X$.

Comment. It is clear that these cycles persists in a codimension-1 submanifold $\mathcal{N} \subset \mathfrak{X}^r(M^3)$.

Let $\{X_\mu\} \subset \mathcal{U}_X$ be a one-parameter family of vector fields such that $X = X_0 \in \mathcal{N}$ and $\{X_\mu\}$ is transversal to $\mathcal{N}$ at $\mu = 0$. We can assume , for $\mu \leq 0$, that $\Upsilon(X_\mu)$ is a wandering orbit.

Now we define a relative neighborhood, U_μ, associated to the cycle. In the first case U_μ is formed by: a linearizing neighborhood, U_0 , of the singularity; a neighborhood, U_1 , of the periodic orbit generated by a transversal section where there are linearizing coordinates for the Poincaré map; a neighborhood of a compact part of the orbit $\Theta(X)$ joining U_0 with U_1, and a neighborhood of a compact part of $\Upsilon(X)$, V_μ, whose points are located at a distance at most $K\mu$, $K \geq 0$, from the stable manifold of the periodic orbit. In the second case we choose a linearizing neighborhood U_0 of the singularity, and a neighborhood of a compact part of Υ_μ, V_μ ,formed by the points that are located at a distance at most $K\mu$ from the stable manifold of the singularity.

Let $H_\mu = \{-\mu \leq \nu \leq \mu,\ \Gamma(X_\nu, U_\mu)$ is a hyperbolic set$\}$, under the previous conditions we have the following

Theorem.

$$\lim_{\mu \to 0} \frac{m(H_\mu)}{\mu} = 1$$

that is: prevalence of hyperbolicity in a relative neighborhood of a complex singular cycle.

We remark that this is a surprising result since all the evidence (see [PRV],[CL],[G]) indicated that, in this situation, the non-hyperbolic systems were prevalent. The important point here is that we restrict the dynamic to a relative neighborhood of the cycle.

We point out that San Martin in [SM] introduces some basic techniques that we use here.

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2. Proof of the Theorem

2.1. Geometry of the first return map

In the sequel we will give a proof of the Theorem in the first case. The second case follows in a similar way.

Let $X \in \mathfrak{X}^r(M^3)$ be a C^r-vector field having a complex singular cycle and $\mathcal{U}_X \subset \mathfrak{X}^r(M^3)$ be a small neighborhood of X.

Note. After a linear change of coordinates, we obtain $b(Y) > 0$.

Let (x_1, x_2, x_3) be C^2-linearizing coordinates, for the vector field $Y \in \mathcal{U}_X$, in a neighborhood of the singularity $\sigma_0(Y)$. In these coordinates Y

has the following form:

$$(Y) := \begin{cases} \dot{x}_1 = a(Y)x_1 - b(Y)x_2 \\ \dot{x}_2 = b(Y)x_1 + a(Y)x_2 \qquad \forall |(x_1, x_2, x_3)| \leq 2. \\ \dot{x}_3 = c(Y)x_3 \end{cases}$$

Let us consider the Poincaré maps defined by the Y-flow on the sections

$$\Sigma_0 = \{(0, x_2, x_3); 0 \leq x_2 \leq 1, |x_3| \leq 1\}$$

and

$$\Sigma_1 = \{(x_1, x_2, 1); |x_1| \leq 1; |x_2| \leq 1\}.$$

$$\pi_{L,Y} : H_0 = \{(0, x_2, x_3) \in \Sigma_0;\ 0 < x_2 \leq 1,\ |x_3| \leq \sigma(Y)^{-1}\} \to \Sigma_0$$

and

$$\pi_{0,Y} : H_1 = \{(0, x_2, x_3) \in \Sigma_0;\ 0 < x_2 \leq 1,\ \sigma(Y)^{-1} \leq x_3 \leq 1\} \to \Sigma_1$$

given by the equations:

$$\pi_{L,Y}(x_2, x_3) = (\lambda(Y)x_2, \sigma(Y)x_3)$$

and

$$\pi_{0,Y}(x_2, x_3) = (-x_3^{-\frac{a(Y)}{c(Y)}} x_2 \sin(\ln(x_3^{-\frac{b(Y)}{c(Y)}})), x_3^{-\frac{a(Y)}{c(Y)}} x_2 \cos(\ln(x_3^{-\frac{b(Y)}{c(Y)}})));$$

where

$$\lambda(Y) = e^{2\pi \frac{a(Y)}{b(Y)}} \quad \text{and} \quad \sigma(Y) = e^{2\pi \frac{c(Y)}{b(Y)}}.$$

Let $Q_1 \subset M^3$ be a transversal section at $q_1(X) \in \sigma_1(X)$, and let (x, y); $|x| \leq 2, |y| \leq 2$ be C^2-linearizing coordinates for the Poincaré map $P(Y)$ associated to the periodic orbit, i.e.

$$P(Y)(x, y) = (\rho(Y)x, \xi(Y)y).$$

We may suppose that $0 < \rho(Y) < 1$ and $\xi(Y) > 1$. We will define

$$p(Y) = (x(\Upsilon), y(\Upsilon)) \in \Upsilon(Y) \cap Q_1$$

$$(\text{resp. } q(Y) = (0, y(\Theta)) \in \Theta(Y) \cap Q_1 \ ),$$

the first (resp. the last) intersection such that $x(\Upsilon) \leq 1$ (resp. $y(\Theta) \leq 1$).

Clearly we have defined a Poincaré map $\pi_{1,Y} : \Sigma_1 \to Q_1$ which is a C^2-diffeomorphism. Obviously $\pi_1(0,0) = p(\Upsilon)$. Moreover, we have defined a C^2 Poincaré map $\pi_{2,Y} : R(Y) \subset Q_1 \to \Sigma_0$; $\pi_{2,Y} = (C(Y,x,y), D(Y,x,y))$, where $R(Y)$ is an horizontal strip containing the connected component of $W^s(\sigma_0(Y)) \cap Q_1$ that contains $q(Y)$. Here a horizontal strip is a closed set, $R \subset Q_1$, bounded (in Q_1) by two disjoint continuous curves connecting the vertical lines $\{(-1,y); |y| \leq 1\}$ and $\{(1,y); |y| \leq 1\}$.

Changing, if necessary, linearizing coordinates we can assume that:

i. $\pi_{1,Y}(x_1,x_2) = (A(Y,x_1,x_2), B(Y,x_1,x_2))$; where

$$\frac{\partial B}{\partial x_2}(Y,0,0) = 0 \quad \text{and} \quad \frac{\partial B}{\partial x_1}(Y,0,0) < 0.$$

Therefore,

$$\frac{\partial A}{\partial x_2}(Y,0,0) \neq 0.$$

ii. $A(Y,0,0) = x(\Upsilon) = c_0, \rho(Y) < c_0 < 1$, $q(Y) = (0, y(\Theta)) = (0,1)$ and $\pi_{2,Y}(q(Y)) = (C(Y,q(Y)), D(Y,q(Y))) = (c_1,0), \lambda(Y) < c_1 < 1$.

iii. The transversality condition implies that

$$\frac{\partial D}{\partial y}(Y;x,y) \neq 0, (x,y) \in R(Y), Y \in \mathcal{U}_X.$$

Without loss we will assume that $\frac{\partial D}{\partial y}(Y,x,y) > 0$ and that, if necessary changing $\pi_{2,Y}$ by a composition $\pi_{2,Y} \circ \pi^j_{L,Y}$ some $j > 0$,the image of the connected component of $W^u(\sigma_1(Y)) \cap Q_1$ that contains $q(Y)$ is a nearly vertical line.

iv. We can take the curves that define the vertical strip ,$R(Y)$,by the implicit equations $D(Y,x,y) = 0$ and $D(Y,x,y) = d$.

The Implicit Function Theorem on Banach spaces implies that the condition $x(\Upsilon) = 0$ defines a C^2 codimension-one submanifold, which is just $\mathcal{N} \subset \mathcal{U}_X$.

We observe that $Y \in \mathcal{U}_X$ implies that $\sigma_1(Y)$ has homoclinic orbits and therefore Y does not have simple dynamics.

Let $U \supset \Gamma$ be a neighborhood of the complex singular cycle Γ. Since $\Gamma(Y,U)$ is the saturation of $\Gamma(Y,U) \cap Q_1$ by the flow Y_t, and $\Gamma(Y,U) \cap Q_1$ is the chain recurrent set of the first return map F_Y associated to the section Q_1, it is necessary to describe the dynamics of F_Y to understand the dynamics of Y on $\Gamma(Y,U)$ (see figure 3).

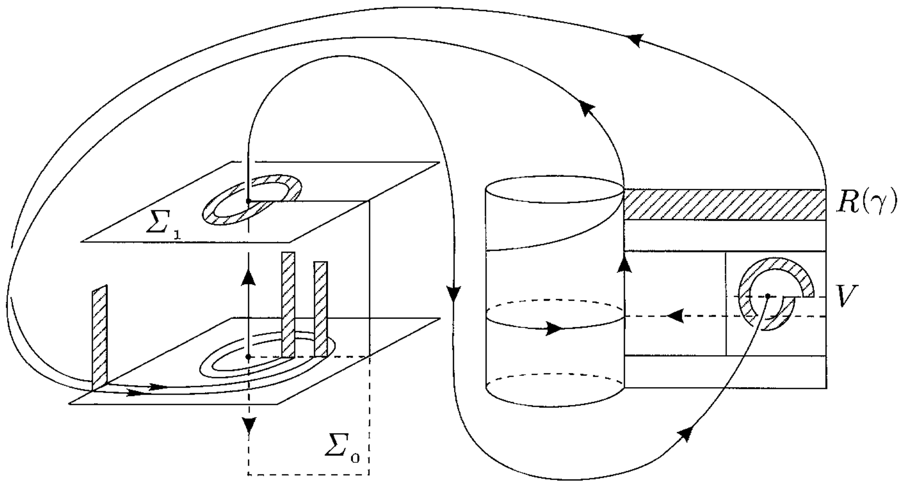

Figure 3: The geometry of the first return map.

Let define

$$V = \{(x, y) \in Q_1; |x - c_0| \le D\}$$

and

$$R_j(Y) = P(Y)^{-j}(R(Y)) \cap V.$$

Let $\mu_0 > 0$ be a small number and $V_{\mu_0} \subset Q_1$ be the relative neighborhood given by:

$$V_{\mu_0} = \{(x, y) \in Q_1 \ : \ |x - c_0| \le \delta\,; |y| \le \mu_0\ \}.$$

Let $m_0 \in \mathbb{N}$ be the greatest integer such that $\xi^{-m_0}(Y) > \mu_0$.

It is clear that a first return map is defined on $\bigcup_{j \ge m_0} R_j(Y)$. In fact, given $(x, y) \in R_j(Y)$ we have

$$P(Y)^j(x, y) \in R(Y); \pi_{2,Y}(P(Y)^j(x, y)) \in \Sigma_0,$$

there exist $n \in \mathbb{N}$ such that

$$\pi_{L,Y}^n(\pi_{2,Y}(P(Y)^j(x, y))) \in H_1;$$
$$\pi_{0,Y}(\pi_{L,Y}^n(\pi_{2,Y}(P(Y)^j(x, y)))) \in \Sigma_1;$$
$$\pi_{1,Y}(\pi_{0,Y}(\pi_{L,Y}^n(\pi_{2,Y}(P(Y)^j(x, y))))) \in Q_1.$$

That is : for $(x, y) \in R_j(Y)$ the first return map, F_Y , is given by :

$$F_Y(x, y) \ = \ \pi_{1,Y} \circ \pi_{0,Y} \circ \pi_{L,Y}^n \circ \pi_{2,Y} \circ P(Y)^j(x, y)\ .$$

In the sequel we will deal with a C^∞ one-parameter family of vector fields $\{X_\mu\} \subset \mathcal{U}_X$ such that $X = X_0$ and $\{X_\mu\}$ is transversal to $\mathcal{N}$ at X. We change the Y- dependence notations for μ. After a new parameterization we have that $\pi_{1,\mu}(0,0) = p(\mu) = (A_\mu(0,0), B_\mu(0,0)) = (c_0, \mu)$.

2.2. Bounds for the locus of the chain recurrent set

Let μ be a parameter value such that $|\mu| \leq \mu_0$, μ_0 fixed. Denote by $V^1_{\mu_0}(\mu)$ the preimage $\pi^{-1}_{1,\mu}(V_{\mu_0})$. Without loss we will assume that

$$\pi_{1,\mu}(\{(0,x);\ |x| \leq 1\} \subset \{(x\,,\mu); 0 < x < 1\}.$$

Denote by $V_d(\mu)$ the image $\pi_{2,\mu}(R(\mu))$. It is clear that :

$$V_d(\mu) \subset \{(0\,,x_2\,,x_3)\ : |\,x_2 - c_1\,| \leq K\ ; 0 \leq x_3 \leq d\,.\}$$

Let $n_1 \in \mathbb{N}$ be the first integer such that: $\sigma_\mu^{-1} \leq \sigma_\mu^{n_1} d < 1$. An easy computation will show the following result.

Lemma 1. *The set* $D_n(\mu) = \{(\lambda^n_\mu x_2\,, x_3\,) \in H_0\}$ *such that*

$$|\,x_2 - c_1\,| \leq K\ ; \textit{and}$$
$$-2C\mu_0 \leq -x_3^{-\frac{a(\mu)}{c(\mu)}} \lambda^n_\mu x_2 \sin(\ln(x_3^{-\frac{b(\mu)}{c(\mu)}})) \leq 2C\mu_0$$

is contained in

$$\{(\lambda^n_\mu x_2\,, x_3)\ ; x_3 \in [\sigma_\mu^{-1}, \sigma_\mu^{-1} + \tilde{C}\mu_0^\gamma] \cup [\sigma_\mu^{-\frac{1}{2}} - \tilde{C}\mu_0^\gamma, \sigma_\mu^{-\frac{1}{2}} + \tilde{C}\mu_0^\gamma] \cup \cup\, [1 - \tilde{C}\mu_0^\gamma, 1]\},$$

for $n_1 \leq n \leq n_2 - 1$. *Here* n_2 *is the greatest integer which satisfies* $\lambda_\mu^{-n}\mu_0 \leq \mu_0^\gamma$; $\tilde{C}$ *is some positive constant and* $0 < \gamma < 1$. *For* $n \geq n_2$ *we have*

$$D_n(\,\mu\,) \subset \{\,(\lambda^n_\mu x_2\,, x_3\,)\ ; |\,x_2 - c_1| \leq K\,, \sigma_\mu^{-1} \leq x_3 \leq 1\}.$$

Now, let

$$C_n(\mu) = \pi^{-n}_{L,\mu}(D_n(\mu)) \cap [c_1 - K, c_1 + K] \times [0, d]$$

for $n_1 \leq n \leq n_2 - 1$ and

$$C_0(\mu) = [c_1 - K, c_1 + K] \times [0\,, \sigma_\mu^{-n_2}]\,.$$

It easy to see that given $\epsilon > 0$, the set

$$\bigcup_{n=n_1}^{n_2-1} C_n(\mu) \cup C_0(\mu)$$

is recovered by

$$(-\frac{Ln\epsilon}{Ln\sigma_\mu} - n_1 + 2)$$

boxes of the type $[c_1 - K, c_1 + K] \times [x_3 - \epsilon, x_3 + \epsilon]$ where x_3 is some point of the sequence

$$\{\sigma_\mu^{-n}, \sigma_\mu^{-n}\sqrt{\sigma_\mu^{-1}}\}_{n \geq n_1}.$$

Let $E_n(\mu)$ denote the preimage $\pi_{2,\mu}^{-1}(C_n(\mu))$. For $j \geq m_0$ we let $E_n^j(\mu)$ denote the set $P_\mu^{-j}(E_n(\mu)) \cap V_{\mu_0}$. Define $y(\mu, n)$ and $y_{\frac{1}{2}}(\mu, n)$ by the equations $D(\mu, 0, y(\mu, n)) = \sigma_\mu^{-n}$ and $D(\mu, 0, y_{\frac{1}{2}}(\mu, n) = \sigma_\mu^{-n}\sqrt{\sigma_\mu^{-1}}$, respectively.

Now we establishes a result that we will use in the sequel :

Lemma 2. *Let $\tau : [-\epsilon, \epsilon] \to]0,1[$ be a lipchitz map. Given $\chi \in]0,1[$ there exists $\epsilon_0 > 0$ and $n_0 \in \mathbb{N}$ such that : For any $\mu; |\mu| \leq \epsilon_0$ and $n \in \mathbb{N}$ such that $n \geq n_0$ we have : $|\tau(\mu)^n - \tau(0)^n| \leq (\tau(0))^{n\chi}|\mu|$.*

Proof. See appendix.

Let $m_0 = m(\mu_0)$ be defined by the relations $\xi_{\mu_0}^{-1} < \xi_{\mu_0}^{m_0}\mu_0 \leq 1$. Using lemma 2 and the previous remark a straightforward computation will show that :

Lemma 3. *Given $\epsilon > 0$ and small $\mu_0 > 0$. For any $|\mu| \leq \mu_0$ we have that $\bigcup_{j \geq m_0}(\bigcup_{n=n_1}^{n_2-1} E_n^j(\mu) \cup E_0^j(\mu))$ is covered by $(-\frac{Ln\epsilon}{Ln\sigma_\mu} - n_1 + 2)(-\frac{Ln\epsilon}{Ln\xi_\mu} + K_1)$ boxes of the type $[c_0 - \delta, c_0 + \delta] \times [y - \epsilon\mu_0, y + \epsilon\mu_0]$, where $y \in \{\xi_0^{-j}y(0,n), \xi_0^{-j}y_{\frac{1}{2}}(0,n); n \geq n_1, j \geq m_0\}$.*

2.3. One-dimensional analysis

Let us now consider the maps $f_{n,\mu,x_2}, f_{\mu,x_2} : [\sigma_\mu^{-1}, 1] \to \mathbb{R}$;

$$f_{n,\mu,x_2}(x_3) = \frac{1}{\lambda_\mu^n}[B_\mu \circ \pi_{0,\mu}(\lambda_\mu^n x_2, x_3) - \mu]$$

and

$$f_{\mu,x_2}(x_3) \;=\; -\partial_{x_1} B_\mu(0,0) x_3^{-a/c} x_2 \sin(\ln(x_3^{-b/c})).$$

Since:

$$\begin{aligned}B_\mu(x_1,x_2) = B_\mu(0,0) + \partial_{x_1} B_\mu(0,0)x_1 + \partial_{x_2} B_\mu(0,0)x_2 + \\ + \frac{1}{2}\partial_{x_1x_1} B_\mu(0,0)x_1^2 + \frac{1}{2}\partial_{x_1x_2} B_\mu(0,0)x_1x_2 + \\ + \frac{1}{2}\partial_{x_2x_2} B_\mu(0,0)x_2^2 + r_{3,\mu}(x_1,x_2),\end{aligned}$$

where

$$\lim_{(x_1,x_2)\to(0,0)} \frac{r_{3,\mu}(x_1,x_2)}{||(x_1,x_2)||^2} = 0;$$

we have:

$$\begin{aligned}B_\mu \circ \pi_{0,\mu}(\lambda_\mu^n x_2, x_3) = & B_\mu(-x_3^{-\frac{a}{c}}\lambda_\mu^n x_2 \sin(\ln(x_3^{-\frac{b}{c}})), x_3^{-\frac{a}{c}}\lambda_\mu^n x_2 \cos(\ln(x_3^{-\frac{b}{c}}))) = \\ = & \mu - \partial_{x_1} B_\mu(0,0)x_3^{-a/c}\lambda_\mu^n x_2 \sin(\ln(x_3^{-b/c})) + \\ & + \frac{1}{2}\partial_{x_1x_1} B_\mu(0,0)(x_3^{-a/c}\lambda_\mu^n x_2 \sin(\ln(x_3^{-b/c})))^2 \\ & - \frac{1}{2}\partial_{x_1x_2} B_\mu(0,0)(x_3^{-a/c}\lambda_\mu^n x_2)^2 \sin(\ln(x_3^{-b/c}))\cos(\ln(x_3^{-b/c})) \\ & + \frac{1}{2}\partial_{x_2x_2} B_\mu(0,0)(x_3^{-a/c}\lambda_\mu x_2 \cos(\ln(x_3^{-b/c})))^2 + r_{3,\mu}(\pi_{0,\mu}(\lambda_\mu^n x_2, x_3)).\end{aligned}$$

Hence:

$$\begin{aligned}f_{n,\mu,x_2}(x_3) = & \frac{1}{\lambda_\mu^n}[B_\mu \circ \pi_{0,\mu}(\lambda_\mu^n x_2, x_3) - \mu] = \\ = & -\partial_{x_1} B_\mu(0,0)x_3^{-a/c} x_2 \sin(\ln(x_3^{-b/c})) + \\ & + \frac{\lambda_\mu^n}{2}\partial_{x_1x_1} B_\mu(0,0)(x_3^{-a/c} x_2 \sin(\ln(x_3^{-b/c}))^2 - \\ & - \frac{\lambda_\mu^n}{2}\partial_{x_1x_2} B_\mu(0,0)(x_3^{-a/c} x_2)^2 \sin(\ln(x_3^{-b/c}))\cos(\ln(x_3^{-b/c})) + \\ & + \frac{\lambda_\mu^n}{2}\partial_{x_2x_2} B_\mu(0,0)(x_3^{-a/c} x_2 \cos(\ln(x_3^{-b/c}))^2 + \\ & + \frac{1}{\lambda_\mu^n} r_{3,\mu}(\pi_{0,\mu}(\lambda_\mu^n x_2, x_3));\end{aligned}$$

since $\sigma_\mu^{-1} \leq x_3 \leq 1$ we have $1 \leq x_3^{-b/c} \leq \sigma_\mu^{b/c}$ and $\sigma_\mu^{a/c} \leq x_3^{-a/c} \leq 1$. Thus, we obtain:

$$|f_{n,\mu,x_2}(x_3) - f_{\mu,x_2}(x_3)| \leq C_1\lambda_\mu^n, \;\; \sigma_\mu^{-1} \leq x_3 \leq 1.$$

In a similar way we get:

$$|f'_{n,\mu,x_2}(x_3) - f'_{\mu,x_2}(x_3)| \leq C_2\lambda_\mu^n$$

and

$$|f''_{n,\mu,x_2}(x_3) - f''_{\mu,x_2}(x_3)| \leq C_3\lambda_\mu^n,$$

any $\sigma_\mu^{-1} \leq x_3 \leq 1$.

So, we proved:

Lemma 4. $||f_{n,\mu,x_2}(\cdot) - f_{\mu,x_2}(\cdot)||_{C^2} \leq C_4\lambda_\mu^n$, *any* $n \in \mathbb{N}$; *any* $\mu \in [-\mu_0, \mu_0]$, μ_0 *small and any* $|x_2 - c_1| \leq K$.

Remark 3.

1. The critical points of the map $f_{\mu,x_2}(\cdot)$, $x_3^1(\mu)$ and $x_3^2(\mu)$ respectively, do not depends on the x_2- variable and satisfy $\sqrt{\sigma_\mu^{-1}} < x_3^1(\mu) < 1$ and $x_3^2(\mu) = x_3^1(\mu)\sqrt{\sigma_\mu^{-1}}$. Moreover $f''_{\mu,x_2}(x_3^i(\mu)) \neq 0$, i : $1,2$ (see figure 4).
2. For the critical values we have:

$$f_{\mu,x_2}(x_3^i(\mu)) = -\partial_{x_1}B_\mu(0,0)(x_3^i(\mu))^{-\frac{a}{c}}x_2\sin(\ln((x_3^i(\mu))^{-\frac{b}{c}})); \ i = 1,2.$$

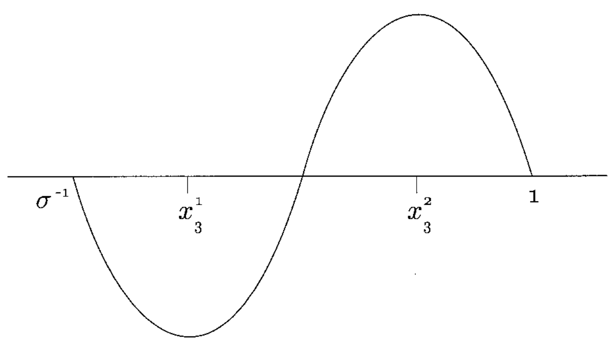

Figure 4: Critical points of the map $f_{\mu,x_2}(\cdot)$.

2.4. Relative neighborhoods and critical points

Let us now compute the critical points of the maps $B_\mu \circ \pi_{0,\mu}(\lambda_\mu^n x_2, x_3)$; $\sigma^{-1} \leq x_3 \leq 1$, x_2 fixed.

Since $B_\mu \circ \pi_{0,\mu}(\lambda_\mu^n x_2, x_3) = \lambda_\mu^n f_{n,\mu,x_2}(x_3) + \mu$, we have that the critical points of the map $B_\mu \circ \pi_0(\lambda_\mu^n x_2, \cdot)$ are the same as for the map $f_{n,\mu,x_2}(\cdot)$.

Let $x_3^i(n,\mu),\ i=1,2$ be the critical points of the map $f_{n,\mu,x_2}(\cdot)$ (they are independent of x_2).

Since

$$|f'_{n,x_2,\mu}(x_3^i(n,\mu)) - f'_{\mu,x_2}(x_3^i(n,\mu))| \leq C_4\lambda_\mu^n,$$

we have that $|f'_{\mu,x_2}(x_3^i(n,\mu))| \leq C_4\lambda_\mu^n$.

From the other side, the equality :

$$|f'_{\mu,x_2}(x_3^i(n,\mu)) - f'_{\mu,x_2}(x_3^i(\mu))| = |f''_{\mu,x_2}(\tilde{x}_3^i)|\ |x_3^i(n,\mu) - x_3^i(\mu)|,$$

implies :

$$|f''_{\mu,x_2}(\tilde{x}_3^i)|\ |x_3^i(n,\mu) - x_3^i(\mu)| = |f'_{\mu,x_2}(x_3^i(n,\mu))| \leq C_4\lambda_\mu^n,$$

that is:

$$|x_3^i(n,\mu) - x_3^i(\mu)| \leq C_5\lambda_\mu^n.$$

Since

$$|f_{n,\mu,x_2}(x_3^i(n,\mu)) - f_{\mu,x_2}(x_3^i(n,\mu))| \leq C_4\lambda_\mu^n,$$

we get:

$$\begin{aligned}
|f_{n,\mu,x_2}(x_3^i(n,\mu)) - f_{\mu,x_2}(x_3^i(\mu))| &\leq |f_{n,\mu,x_2}x_3^i(n,\mu) - f_{\mu,x_2}x_3^i(n,\mu)| \\
&\quad + |f_{\mu,x_2}x_3^i(n,\mu) - f_{\mu,x_2}(x_3^i)| \\
&\leq K\lambda_\mu^n + |f'_{\mu,x_2}(\overline{x_3})||x_3^i(n,\mu) - x_3^i(\mu)| \\
&\leq C_4\lambda_\mu^n + C_6\lambda_\mu^n \\
&= C_7\lambda_\mu^n
\end{aligned}$$

that is:

$$|f_{n,\mu,x_2}(x_3^i(n,\mu)) - f_{\mu,x_2}(x_3^i(\mu))| \leq C_7\lambda_\mu^n.$$

From the equality:

$$B_\mu \circ \pi_{0,\mu}(\lambda_\mu^n x_2, x_3^i(n,\mu)) = \mu + \lambda_\mu^n f_{n,\mu,x_2}(x_3^i(n,\mu),$$

we obtain:

$$\begin{aligned}
&|B_\mu \circ \pi_{0,\mu}(\lambda_\mu^n x_2, x_3^i(n,\mu)) - \mu - \lambda_\mu^n f_{\mu,x_2}(x_3^i(\mu))| \leq \\
&\leq \lambda_\mu^n |f_{n,\mu,x_2}(x_3^i(n,\mu)) - f_{\mu,x_2}(x_3^i(\mu))| \leq C_7\lambda_\mu^{2n}.
\end{aligned}$$

Let $y_n^i(\mu, x_2) = B_\mu \circ \pi_{0,\mu}(\lambda_\mu^n x_2, x_3^i(n,\mu))$, we have proved.

Lemma 5. *There are constants* $C_5, C_7 > 0$ *such that*

$$|x_3^i(n,\mu) - x_3^i(\mu)| \leq C_5 \lambda_\mu^n, \tag{3}$$

and

$$|y_n^i(\mu, x_2) - (\mu + \lambda_\mu^n f_{\mu,x_2}(x_3^i(\mu)))| \leq C_7 \lambda_\mu^{2n} \quad (4).$$

for $i = 1, 2$.

Let us now define: $\chi_n^i(\mu, x_2) = \mu + \lambda_\mu^n f_{\mu,x_2}(x_3^i(\mu))$. Since $\lambda = \lambda_\mu$ and

$$f_{\mu,x_2}(x_3^i(\mu)) = -\partial_{x_2} B_\mu(0,0)(x_3^i(\mu))^{-a/c} \sin(\ln(x_3^i(\mu))^{-b/c}))x_2 = t_i(\mu)x_2,$$

we have: $\chi_n^i(\mu, x_2) = \mu + \lambda_\mu^n x_2 t_i(\mu)$, $t_1(\mu) > 0$ and $t_2(\mu) < 0$.

Applying lemma 2 to the map $\tau(\mu) = \lambda_\mu$ we obtain:

$$\lambda_0^n - \lambda_0^{n\chi}|\mu| \leq (\lambda_\mu)^n \leq \lambda_0^n + \lambda_0^{n\chi}|\mu|,$$

therefore, using lemma 5

$$|y_n^i(\mu, x_2) - (\mu + \lambda_0^n t_i(\mu)x_2)| \leq C_7 \lambda_0^{2n} + \lambda_0^{n\chi}|\mu| C_8 x_2.$$

Lemma 6. *Given* $1 > \chi > 0$, *there are* $n_0 \in \mathbb{N}$ *and constants* C_7, C_9, $C_{10} > 0$ *such that for all small* $\mu_0 > 0$,

$$|y_n^i(\mu, x_2) - (\mu + \lambda_0^n t_i(0)c_1)| \leq C_7 \lambda_0^{2n} + \lambda_0^{n\chi} C_9 \mu_0 + C_{10} \lambda_0^n |x_2 - c_1|$$

any $n \geq n_0$, $i = 1, 2$, *and* $|\mu| \leq \mu_0$. *That is: the critical values* $y_n^i(\mu, x_2)$ *are in the* $(C_7 \lambda_0^{2n} + \lambda_0^{n\chi} C_9 \mu_0 + C_{10} \lambda_0^n |x_2 - c_1|)$*-neighborhood of the points* $\mu + \lambda_0^n t_i(0)c_1$.

2.5. Proof of the Theorem

Let $\mu_0 > 0$ be a small parameter value and $V_{\mu_0} \subset Q$ be the relative neighborhood given by

$$V_{\mu_0} = \{(x,y) \in Q; |x - c_0| \leq \delta\ ,\ |y| \leq \mu_0\}.$$

Let $|\mu| \leq \mu_0$ be a parameter value.

Let $(x_0, y_0) \in V_{\mu_0} \cap \Gamma(X_\mu, U_{\mu_0})$ and $(x, y) \in V_{\mu_0} \cap \Gamma(X_\mu, U_{\mu_0})$ be two points related by the equation :

$$(x_0, y_0) = F_\mu(x, y) = \pi_{1,\mu} \circ \pi_{0,\mu} \circ \pi_{L\mu}^n \circ \pi_{2,\mu} \circ P_\mu^m(x, y);$$

where $n \geq n_1$, $m \geq m_0$.

Clearly, for any $n \geq n_1$ such that $\lambda_0^n \geq 3\mu_0$, the angle, ∇, between the images of nearly vertical lines trough (x, y) and the horizontal line trough (x_0, y_0) is bigger than a constant $C >> 0$ (see figure 5).

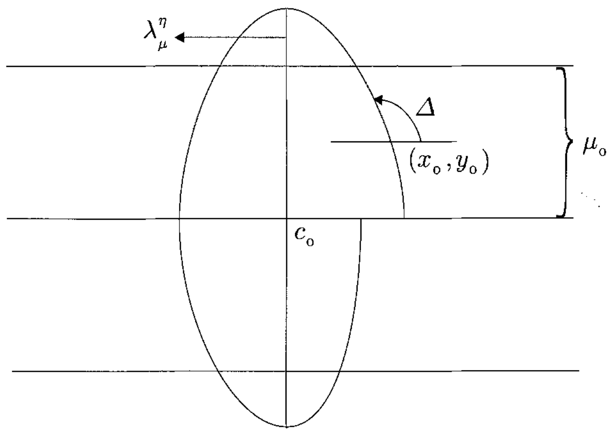

Figure 5: The angle ∇.

Let $n \geq n_1$ be an integer such that $\lambda_0^n < 3\mu_0$.

In this situation, given $\epsilon > 0$ we will choose parameter values in such a way that $\nabla > C(\epsilon) > 0$.

First, we give a precise bound for the locus of the critical values in Lemma 6. In fact, we have $|x - c_1| < C_{11}\mu_0^a$ where

$$0 < \ a \ < \min\{-\frac{Ln(\sigma_\mu)}{Ln(\lambda_\mu)}, -\frac{Ln(\rho_\mu)}{Ln(\xi_\mu)}\}.$$

Hence,

$$|y_n^i(\mu, x_2) - (\mu + \lambda_0^n t_i(0)c_1)| \leq C_{12}\mu_0^{1+t}, i = 0, 1$$

and $t = \min\{a, \chi\}$.

Let

$$C(\mu_0) = \{(\lambda_0)^n t_0(0)c_1, (\lambda_0)^n t_1(0)c_1 \, ; \text{any}\, n\, \text{such that} (\lambda_0)^n < 3\mu_0\}$$

and

$$E(\mu_0) = \{(\xi_0)^{-j} y(0, n), (\xi_0)^{-j} y_{\frac{1}{2}}(0, n) \, ; n \geq n_1, j \geq m_0\}.$$

Denote by $C_\epsilon(\mu_0)$ and $E_\epsilon(\mu_0)$ the $\epsilon\mu_0$-neighborhood of the sets $C(\mu_0)$ and $E(\mu_0)$, respectively.

Let $H_{\mu_0}(\epsilon) = \{\mu \in [-\mu_0 , \mu_0]; d(C_\epsilon(\mu_0) + \mu , E_\epsilon(\mu_0)) \geq \epsilon\mu_0 \}$.

Lemma 7. *For any $\mu_0 > 0$ small enough we have*

$$m([-\mu_0 , \mu_0] \; - \; H_{\mu_0}(\epsilon)) \leq g(\epsilon)\mu_0$$

here $g(\epsilon)$ satisfies $g(\epsilon) \to 0$ as $\epsilon \to 0$.

Proof. It is easy to see that $C_\epsilon(\mu_0)$ can be covered by

$$g_1(\epsilon) = \left(\frac{Ln(\epsilon)}{Ln(\lambda_0)} + C_{12} \right) - \text{intervals}$$

whose length is $\epsilon\mu_0$. From Lemma 3 we know that we can cover $E_{2\epsilon}(\mu_0)$ with

$$g_2(\epsilon) \; = \; \left(-\frac{Ln(2\epsilon)}{Ln(\sigma_\mu)} - n_1 + 2\right)\left(- \frac{Ln(2\epsilon)}{Ln(\xi_\mu)} + K_1 \right) - \text{intervals}$$

whose length is $\epsilon\mu_0$.

Let $\mu \in (H_{\mu_0}(\epsilon))^c$. We have that $d(C_\epsilon(\mu_0) + \mu , E_\epsilon(\mu_0)) < \epsilon\mu_0$, that is, there are $x \; = \; z + \mu , z \in C_\epsilon(\mu_0) \; ; y \in E_\epsilon(\mu_0)$ such that $d(x , y) < \epsilon\mu_0$. Hence, $x \in E_\epsilon(\mu_0)$ and $\mu \in E_{3\epsilon}(\mu_0) - C_\epsilon(\mu_0)$. As a consequence we obtain :

$$m((H_{\mu_0}(\epsilon))^c) \leq g_1(\epsilon) \; g_2(\epsilon) \; 5\epsilon\mu_0.$$

Taking $g(\epsilon) = g_1(\epsilon) \; g_2(\epsilon) \; 5\epsilon$, we get the result.

Lemma 8. *For any $\mu \in H_{\mu_0}(\epsilon)$ the chain recurrent set $\Gamma(X_\mu , U_{\mu_0})$ is hyperbolic.*

Proof. First we observe that the distance between the turning points of the images of the map F_μ and the set $E_\epsilon(\mu_0)$ is at least $\epsilon\mu_0$.

Let us consider the map F_μ. For $(x_0 , y_0) \in V_{\mu_0} \cap \Gamma(X_\mu , U_{\mu_0})$ we have :

$$(x_0 , y_0) \; = F_\mu(x, y) = \pi_1 \circ \pi_0 \circ \pi^n_{L_\mu} \circ \pi_2(\rho^m x, \xi^m y).$$

Putting $(x_2, x_3) = \pi_2(\rho^m_\mu x, \xi^m_\mu y)$ we have:

$$F_\mu(x, y) = \pi_1 \circ \pi_0(\lambda^n_\mu x_2, \sigma^n_\mu x_3) = \pi_1 \circ \pi_0(\lambda^n_\mu x_2, z), \; z = \sigma^n_\mu x_3.$$

Assume x_2 is a fixed value and consider the curve:

$$E = \{(A_\mu \circ \pi_0(\lambda_\mu^n x_2, z), B_\mu \circ \pi_0(\lambda_\mu^n x_2, z));\ \ \sigma_\mu^{-1} \le z \le 1\}.$$

This curve makes an angle,φ, with the horizontal line $y = y_0$, given by:

$$\tan(\varphi) = \frac{\partial_z(B_\mu \circ \pi_0)(\lambda_\mu^n x_2, z)}{\partial_z(A_\mu \circ \pi_0)(\lambda_\mu^n x_2, z)} \ ; z = (\sigma_\mu)^n x_3.$$

Since:

$$\begin{aligned}\partial_z(A_\mu \circ \pi_0)(\lambda_\mu^n x_2, z) &= \partial_{x_1} A_\mu(\pi_0(\lambda_\mu^n x_2, z)) \cdot \lambda_\mu^n x_2 \cdot \\ &\cdot \frac{\partial}{\partial z}(-z^{-\frac{a}{c}} \sin(\ln(z^{-\frac{b}{c}}))) + \partial x_2 A_\mu(\pi_0(\lambda_\mu^n x_2, z)) \cdot \\ &\cdot \lambda_\mu^n x_2 . \cdot \frac{\partial}{\partial}(z^{-\frac{a}{c}} \cos(\ln(z^{-\frac{b}{c}}))) \ ;\end{aligned}$$

we get:

$$|\partial_z(A_\mu \circ \pi_0)(\lambda_\mu^n x_2, z)| \le C_{13}^{-1} \cdot \lambda_\mu^n.$$

So, we obtain
$|\tan\varphi| \ge C_{13} \cdot \frac{|\partial_z(B_\mu \circ \pi_0)(\lambda_\mu^n x_2, z)|}{\lambda_\mu^n} = C_{13} |f'_{n,\mu,x_2}(z)|$.
Let us now consider the map $f_{n,\mu,x_2}(\cdot)$ (see figure 6)

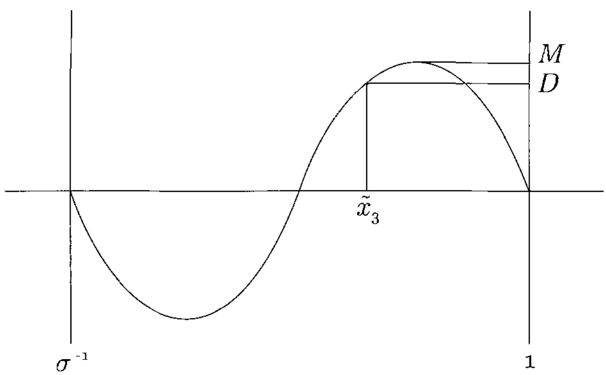

Figure 6.

We observe that :

$$|f'_{n,\mu,x_2}(z)| \ge C_{14}\sqrt{M - D}$$

if $M - D$ is small and

$$|f'_{n,\mu,x_2}(z)| \ge C_{15} > 0$$

otherwise (for instance $\lambda_0^n > 3\mu_0$) any $z \in]\sigma^{-1}, 1[$.

Since $\mu \in H_{\mu_0}(\epsilon)$ our possible election ,for the z-values ,satisfies the condition that the minimal value for $M-D$ is $\lambda_\mu^{-n}\mu_0\epsilon$. Thus we conclude that:

$$|\tan\varphi| \geq C_{14}\sqrt{\lambda_\mu^{-n}\mu_0\epsilon} \geq C_{15}\sqrt{\epsilon}\,.$$

Now we are ready to construct a cone field

$$C_\mu(x,y)\,,(x,y) \in V_{\mu_0} \cap \Gamma(X_\mu\,, U_{\mu_0})\,, \mu \in H_{\mu_0}(\epsilon)$$

such that

(i) $DF_\mu(C_\mu(x,y)) \subset C_\mu(F_\mu(x,y))$;

(ii) there exists $C_{16} > 1$ such that:

a. $||DF_\mu(x,y)(v_1,v_2)|| \geq C_{16}||(v_1,v_2)||\,,(v_1,v_2) \in C_\mu(x,y)$;

b. $||DF_\mu(x,y)(v_1,\, v_2)|| \leq C_{16}^{-1}||(v_1\,,v_2)||$, any $(v_1\,,v_2)$ in the complement of the cone $DF_\mu^{-1}(F_\mu(x\,,y))(C_\mu(F_\mu(x\,,y))$.

For any $(x,y) \in V_{\mu_0} \cap \Gamma(X_\mu\,, U_{\mu_0})$ let $C_\mu(x,y)$ be the cone whose angle with the vertical axis is $\theta_0 = 90 - \frac{C(\epsilon)}{2}$.

For $y > 0$ there are $m \geq m_0$ and $n \geq n_1$ such that

$$F_\mu(x,y) = \pi_{1,\mu} \circ \pi_{0,\mu} \circ \pi_{L\mu}^n \circ \pi_{2,\mu} \circ P_\mu^m(x,y).$$

Applying P_μ^m to the cone $C_\mu(x,y)$ we have that the angle θ_m, with the vertical line, satisfy:

$$\tan\theta_m = (\frac{\rho_\mu}{\xi_\mu})^m \tan(\theta_0), \quad m \geq m_0.$$

Since $\pi_{2,\mu}$ is a diffeomorphism which send vertical lines (in (x,y)) into nearly vertical lines (in (x_2,x_3)), we have that the image of this cone (by the map $D(\pi_{2,\mu})$) is included in a cone whose angle with the vertical lines,β_0 ,is near the angle, α_m ,that the image of this cone(by the map $D(\pi_{2,\mu} \circ P_\mu^m)$) makes with the vertical lines. Here α_m satisfies:

$$\tan\alpha_m = C_{17}\tan\theta_m.$$

Therefore the angle,β_n ,of the image $D(\pi_{L,\mu}^n \circ \pi_{2,\mu} \circ P_\mu^m)$, with the vertical lines satisfy

$$\tan\beta_n = (\frac{\lambda}{\sigma})^n \tan\beta_0.$$

Now it is not difficult to see that the map $D(\pi_{1,\mu} \circ \pi_{0,\mu})$ send this cone into one located in the tangent space whose angle is of the order of $\lambda_0^n(number)$. If $\lambda_0^n \geq 3\mu_0$ then clearly $DF_\mu(x,y)C_\mu(x,y) \subset C_\mu(F_\mu(x,y))$. If $\lambda_0^n \leq 3\mu_0$ then, for small μ_0 we get the same result. This, together with the expansion of the vectors in the cone, completes the prove of the first part of the proof of the hyperbolicity. The second part follows in a similar way.

The Theorem in the introduction is a consequence of Lemma 7 and Lemma 8.

3. Appendix

Here we give a proof of the following result:

Lemma 2. *Let $\tau : [-\epsilon, \epsilon] \to]0,1[$ be a Lipchitz map. Given $\chi \in]0,1[$ there are $\epsilon_0 > 0$ and $m_0 \in \mathbb{N}$ such that:*

For any $\mu; |\mu| \leq \epsilon_0$ and $m \in \mathbb{N}$ such that $m \geq m_0$ we have:

$$|(\tau(\mu))^m - (\tau(0))^m| \leq (\tau(0))^{m\chi}|\mu|.$$

Proof.

$$|(\tau(\mu))^m - (\tau(0))^m| = (\tau(0))^m|\left(\frac{\tau(\mu)}{\tau(0)}\right)^m - 1|$$

$$\left|\frac{\tau(\mu)}{\tau(0)} - 1\right| = (\tau(0))^{-1}|\tau(\mu) - \tau(0)| \leq \frac{K|\mu|}{\tau(0)} = \tilde{K}|\mu|.$$

So, we have :

$$(1 - \tilde{K}|\mu|)^m \leq \left(\frac{\tau(\mu)}{\tau(0)}\right)^m \leq (1 + \tilde{K}|\mu|)^m$$

from this we get

$$(1 - \tilde{K}|\mu|)^m - 1 \leq \left(\frac{\tau(\mu)}{\tau(0)}\right)^m - 1 \leq (1 + \tilde{K}|\mu|)^m - 1.$$

Let $\eta(\nu) = (1 + \tilde{K}\nu)^m - 1$. We have $\eta(0) = 0$ and

$$\eta'(\nu) = \tilde{K}m(1 + \tilde{K}\nu)^{m-1}.$$

By the mean value theorem we get:

$$\eta(\nu) - \eta(0) = \eta'(\tilde{\nu}) \cdot \nu = \tilde{K}m(1 + \tilde{K}\tilde{\nu})^{m-1}\nu.$$

Hence,

$$(1 + \tilde{K}\nu)^m - 1 = \tilde{K}m(1 + \tilde{K}\tilde{\nu})^{m-1}\nu$$

Putting $l(\nu) = (1 - \tilde{K}\nu)^m - 1$, we have $l(0) = 0$, $l'(\nu) = m\tilde{K}(1 - \tilde{K}\nu)^{m-1}\nu$ and then:

$$l(\nu) - l(0) = l'(\tilde{\nu}) \cdot \nu,$$

that is

$$(1 - \tilde{K}\nu)^m - 1 = -m\tilde{K}(1 - \tilde{K}\tilde{\nu})^{m-1}\nu.$$

The second part follows in a similar way. Taking $\nu = |\mu|$ we obtain

$$(1 - \tilde{K}|\mu|)^m - 1 = -m\tilde{K}(1 - \tilde{K}|\tilde{\mu}|)^{m-1}|\mu|.$$

By choosing $\tilde{\mu}$ or $\overline{\mu}$ we have:

$$-m\tilde{K}(1 - \tilde{K}|\tilde{\mu}|)^{m-1}|\mu| \leq \left(\frac{\tau(\mu)}{\tau(0)}\right)^m - 1 \leq \tilde{K}m(1 + \tilde{K}|\tilde{\mu}|)^{m-1}|\mu|.$$

That is:

$$\left|\left(\frac{\tau(\mu)}{\tau(0)}\right)^m - 1\right| \leq \tilde{K}m(1 + \tilde{K}|\tilde{\mu}|)^{m-1}|\mu|.$$

Then

$$(\tau(0))^m \left|\left(\frac{\tau(\mu)}{\tau(0)}\right)^m - 1\right| \leq \tilde{K}m(\tau(0))^m(1 + \tilde{K}|\tilde{\mu}|)^m|\mu|;$$

since $\tau(0) \in]0,1[$, for small $|\mu|$, we have $\tau(0)(1 + \tilde{K}|\tilde{\mu}|) < 1$.

Now, we choose $n_0 \in \mathbb{N}$ such that

$$\tilde{\tilde{K}}m(\tau(0))^m(1 + \tilde{K}|\tilde{\mu}|)^m \leq (\tau(0))^{m\chi}.$$

for any $m \geq m_0$.

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